§ The Bellman Function for $\boxed{S_{\mathcal{D}} : L^2(\mathbb{R}) \to L^{2,\infty}(\mathbb{R})}$

→ We derive the exact Bellman function for the weak L^2 bound for the dyadic square function. (This is a boundary case of the corresponding weighted estimate $S_{\mathcal{D}} : L^2(w) \to L^{2,\infty}(w)$ for $w \in A_2$ - sharp constant unknown).

→ Remark: The square function trivially satisfies this weak bound:

We are after

$$\|S_{\mathcal{D}} f\|_{2,\infty} \leq C \|f\|_2 \quad \text{, i.e. } \left|\{x : S_{\mathcal{D}} f(x) > \lambda\}\right| \leq \frac{C^2}{\lambda} \|f\|_2^2$$

or simpler $\left|\{x : S_{\mathcal{D}}^2 f(x) > \lambda\}\right| \leq \frac{C^2}{\lambda} \|f\|_2^2$

This is satisfied w/ constant $C = 1$:

$$\|S_{\mathcal{D}} f\|_{2,\infty} \leq \|S_{\mathcal{D}} f\|_2 = \|f\|_2$$

→ Truncated Dyadic Square Function:

$$\boxed{S_J \varphi(x) := \left(\sum_{I \subset J} (\varphi, h_I)^2 \frac{\mathbb{1}_I(x)}{|I|} \right)^{1/2}}$$

=> Localize the problem

=> need to show something like: $\left|\{x \in J : S_J^2 \varphi(x) > \lambda\}\right| \leq \frac{C}{\lambda} \int_J \varphi^2$

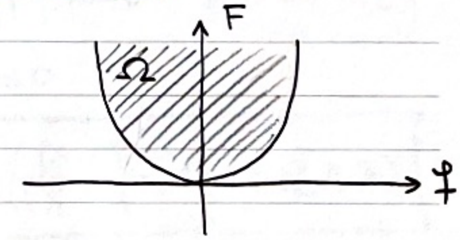
(divide by $|J|$, makes it scale invariant): $\frac{1}{|J|} \left|\{x \in J : S_J^2 \varphi(x) > \lambda\}\right| \leq \frac{C}{\lambda} \langle \varphi^2 \rangle_J$

The Bellman Function of the Problem:

$$\boxed{\mathcal{B}(\mathfrak{f}, F, \lambda) := \sup \frac{1}{|J|} \left|\{x \in J : S_J^2 \varphi(x) > \lambda\}\right|}$$

, where supremum is over all test functions φ, supported in J, and satisfying

$$\langle \varphi \rangle_J = \mathfrak{f}; \quad \langle \varphi^2 \rangle_J = F.$$



→ $\mathcal{B}$ does not depend on the choice of J;

→ Domain: $\boxed{\Omega := \left\{(\mathfrak{f}, F, \lambda) : \mathfrak{f}^2 \leq F, \ \lambda > 0\right\}}$

$$\mathfrak{f}^2 = \langle \varphi \rangle_J^2 \leq \langle \varphi^2 \rangle_J \text{ by Hölder}$$

→ Range: $\boxed{0 \leq \mathcal{B} \leq 1}$

Obstacle Condition: $\boxed{\text{If } \lambda < F - f^2, \text{ then } B = 1.}$

Given any f, let the test function $\varphi := f + c\sqrt{|J|}\, h_J \Rightarrow \langle\varphi\rangle_J = f$

$$\varphi = \begin{cases} f + c & \text{on } J_+ \\ f - c & \text{on } J_- \end{cases} \Rightarrow \langle\varphi^2\rangle_J = \frac{1}{|J|}\left(\int_{J_+}(f+c)^2 + \int_{J_-}(f-c)^2\right)$$

$$= \frac{1}{|J|}\,\frac{|J|}{2}\left(2f^2 + 2c^2\right) = f^2 + c^2$$

Choose c such that $F = f^2 + c^2 \Rightarrow \langle\varphi^2\rangle_J = F$

$\Rightarrow \varphi$ is admissible for $B(f, F, \lambda)$ and $S_J^2\varphi = c^2|J|\frac{\mathbb{1}_J}{|J|} = (F - f^2)\mathbb{1}_J$

$\Rightarrow$ given any $\lambda < F - f^2$, $S_J^2\varphi > \lambda$ everywhere on $J \Rightarrow B \equiv 1$.

Boundaries: $\boxed{B(f, F, \lambda) = 0, \ \forall f^2 = F}$

If $f^2 = F$, any admissible function φ must satisfy $\langle\varphi\rangle_J^2 = \langle\varphi^2\rangle_J$, or equality in Hölder $\Rightarrow \varphi$ must be constant $\Rightarrow \varphi \equiv f$, but then $S_J\varphi \equiv 0.$

Homogeneities: Let φ be admissible for $B(f, F, \lambda)$, i.e. $\langle\varphi\rangle_J = f$; $\langle\varphi^2\rangle_J = F$.

- Take $\tilde\varphi = c\varphi$ for a scalar $c \Rightarrow \left.\begin{array}{l}\langle\tilde\varphi\rangle_J = cf \\ \langle\tilde\varphi^2\rangle_J = c^2F\end{array}\right\} \Rightarrow \tilde\varphi$ admissible for $B(cf, c^2F, \underset{c^2\lambda}{?})$

and

$$S_J^2\tilde\varphi(x) = c^2 S_J^2\varphi(x)$$

$$\frac{1}{|J|}\left|\{x \in J : S_J^2\tilde\varphi(x) = c^2S_J^2\varphi(x) > c^2\lambda\}\right| = \frac{1}{|J|}\left|\{x \in J : S_J^2\varphi(x) > \lambda\}\right|$$

$$\boxed{B(cf, c^2F, c^2\lambda) = B(f, F, \lambda)} \quad \forall c \neq 0$$

$$\xrightarrow{c = 1/\sqrt{\lambda}} \boxed{B\left(\frac{f}{\sqrt{\lambda}}, \frac{F}{\lambda}, 1\right) = B(f, F, \lambda)}$$

- Take $\tilde\varphi = \varphi + c$ for a scalar c

$$\left.\begin{array}{l}\langle\tilde\varphi\rangle_J = c + f \\ \langle\tilde\varphi^2\rangle_J = \langle\varphi^2 + 2c\varphi + c^2\rangle_J = F + 2cf + c^2\end{array}\right\} \Rightarrow \text{adm. for } B(c+f, F+2cf+c^2, \lambda) \text{ and } S_J^2(\varphi + c) = S_J^2\varphi$$

$$\Rightarrow \boxed{B(c+f, F+2cf+c^2, \lambda) = B(f, F, \lambda)}$$

$$\xrightarrow{c = -f} \boxed{B(0, F - f^2, \lambda) = B(f, F, \lambda)}$$

Put together: $B(f, F, \lambda) = B(0, F - f^2, \lambda) = B\left(0, \frac{F-f^2}{\lambda}, 1\right)$

$$B(f,F,\lambda) = B\left(0, \frac{F-f^2}{\lambda}, 1\right) =: \underline{\underline{\theta(\tau)}}, \quad \boxed{\tau = \frac{F-f^2}{\lambda}}$$

Reframe $\underline{OC}$: if $1 < \frac{F-f^2}{\lambda}$, then $B = \theta = 1$ $\Rightarrow$ if $\tau > 1$, $\theta(\tau) = 1$

Reframe boundaries:

$$B(f,F,\lambda) = 0, \text{ if } f^2 = F \Rightarrow \theta(0) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} B(f,F,\lambda) = \theta(\tau)\ ;\ \tau = \frac{F-f^2}{\lambda}\ ;\ \tau \geq 0 \\ \bullet\ \theta(\tau) = 1,\ \forall \tau > 1 \\ \bullet\ \theta(0) = 0 \end{array}}$$

→ **Remark**: B is even in the first variable (f):

φ admissible for $B(f,F,\lambda) \Leftrightarrow \langle\varphi\rangle_J = f,\ \langle\varphi^2\rangle_J = F$

$\Leftrightarrow -\varphi$ is admissible for $B(-f,F,\lambda)$

and $S_J\varphi = S_J(-\varphi)$

$$\Rightarrow \boxed{B(f,F,\lambda) = B(-f,F,\lambda)}$$

→ **Remark**: B is decreasing in λ

$\lambda_1 < \lambda_2$

$$\frac{1}{|J|}\left|\{x \in J : S_J^2\varphi(x) > \lambda_2\}\right| \leq \frac{1}{|J|}\left|\{x \in J : S_J^2\varphi(x) > \lambda_1\}\right| \quad \forall \varphi \text{ admissible}$$

$$\Rightarrow B(f,F,\lambda_2) \leq B(f,F,\lambda_1).$$

→ **Main Inequality:** $\boxed{\begin{array}{l} \mathcal{B}\left(f, F, \lambda+\left(\frac{f_+-f_-}{2}\right)^2\right) \geq \frac{1}{2}\Big(\mathcal{B}(f_+, F_+, \lambda)+\mathcal{B}(f_-, F_-, \lambda)\Big) \\ \text{for all points in the domain with } f=\frac{1}{2}(f_++f_-) \,\&\, F=\frac{1}{2}(F_++F_-) \end{array}}$

Let $\varphi_\pm$ be supported on $J_\pm$, admissible for $\mathcal{B}(f_\pm, F_\pm, \lambda)$, that almost give the supremum (up to some $\varepsilon>0$):

$$\langle \varphi_\pm \rangle_{J_\pm} = f_\pm \; ; \; \langle \varphi_\pm^2 \rangle_{J_\pm} = F_\pm$$

$$\frac{1}{|J_\pm|}\left|\{x\in J_\pm : S^2_{J_\pm}\varphi_\pm(x)>\lambda\}\right| > \mathcal{B}(f_\pm, F_\pm, \lambda)-\varepsilon \quad (*)$$

Define φ by concatenation: $\varphi := \varphi_- \mathbb{1}_{J_-} + \varphi_+ \mathbb{1}_{J_+}$. Then φ is supported in J, satisfies $\langle \varphi \rangle_J = \frac{1}{2}(f_-+f_+) = f$ and $\langle \varphi^2 \rangle_J = \frac{1}{2}(F_-+F_+) = F$, and

$$S_J^2\varphi = (\varphi, h_J)^2 \frac{\mathbb{1}_J}{|J|} + S^2_{J_-}\varphi_- + S^2_{J_+}\varphi_+ \qquad (\varphi, h_J) = \frac{1}{\sqrt{|J|}}\left(\int_{J_+}\varphi_+ - \int_{J_-}\varphi_-\right)$$

$$= \left(\frac{f_+-f_-}{2}\right)^2 \mathbb{1}_J + S^2_{J_-}\varphi_- + S^2_{J_+}\varphi_+ \qquad = \sqrt{|J|}\,\frac{1}{2}(f_+-f_-)$$

$\Rightarrow \varphi$ is admissible for $\mathcal{B}\left(f, F, \lambda+\left(\frac{f_+-f_-}{2}\right)^2\right)$ and so:

$$\mathcal{B}\left(f, F, \lambda+\left(\frac{f_+-f_-}{2}\right)^2\right) \geq \frac{1}{|J|}\left|\left\{x\in J : S_J^2\varphi(x) > \lambda+\left(\frac{f_+-f_-}{2}\right)^2\right\}\right|$$

$$= \frac{1}{2|J_-|}\left|\{x\in J_- : S^2_{J_-}\varphi_-(x)>\lambda\}\right| + \frac{1}{2|J_+|}\left|\{x\in J_+ : S^2_{J_+}\varphi_+(x)>\lambda\}\right|$$

$$\underset{(*)}{>} \frac{1}{2}\Big(\mathcal{B}(f_-, F_-, \lambda)+\mathcal{B}(f_+, F_+, \lambda)\Big)-\varepsilon \quad , \forall \varepsilon>0 \Rightarrow \text{result follows.} \; \blacksquare$$

Remark: Rewrite the MI in the following useful way - by expressing

$$f_- = f-a \; ; \; f_+ = f+a \quad \text{and} \quad F_- = F-b \; ; \; F_+ = F+b \; :$$

$$\mathcal{B}(f, F, \lambda) \geq \frac{1}{2}\Big[\mathcal{B}(f+a, F+b, \lambda-a^2)+\mathcal{B}(f-a, F-b, \lambda-a^2)\Big]$$

- Let $a=0$ above $\Rightarrow \mathcal{B}(f, F, \lambda) \geq \frac{1}{2}\Big(\mathcal{B}(f, F+b, \lambda)+\mathcal{B}(f, F-b, \lambda)\Big) \Rightarrow$ concavity in the 2^{nd} variable
- Let $b=0$ above $\Rightarrow \mathcal{B}(f, F, \lambda) \geq \frac{1}{2}\Big(\mathcal{B}(f+a, F, \lambda-a^2)+\mathcal{B}(f-a, F, \lambda-a^2)\Big)$

  $\geq \frac{1}{2}\Big(\mathcal{B}(f+a, F, \lambda)+\mathcal{B}(f-a, F, \lambda)\Big) \Rightarrow$ concavity in the 1^{st} variable

  b/c $\mathcal{B}$ is decreasing in λ

- Let $f=0$ and $b=0$: $\mathcal{B}(0, F, \lambda) \geq \frac{1}{2}\Big(\mathcal{B}(a, F, \lambda)+\mathcal{B}(-a, F, \lambda)\Big) = \mathcal{B}(a, F, \lambda)$

  $\Rightarrow$ $\mathcal{B}$ has a maximum at $f=0$. (b/c $\mathcal{B}$ is even in 1^{st} var.)